ARITHMETICAL INDEPENDENCE RESULTS USING HIGHER RECURSION THEORY

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Abstract. We extend an independence result proved in [1]. We show that for all n, there is a special set of Π_n sentences $\{\varphi_a\}_{a\in H}$ corresponding to elements of a linear ordering $(H, <_H)$ of order type $\omega_1^{CK}(1+\eta)$. These sentences allow us to build completions $\{T_a\}_{a\in H}$ of PA such that for $a <_H b$, $T_a \cap \Sigma_n \subset T_b \cap \Sigma_n$, with $\varphi_a \in T_a$, $\neg\varphi_a \in T_b$. Our method uses the Barwise–Kreisel Compactness Theorem.

§1. Introduction. We have two goals in this paper. Our first goal is to present some new independence results for first-order Peano Arithmetic (PA). Our second goal is to introduce our method for proving these results, which is grounded on an application of the Barwise–Kreisel Compactness Theorem.

A few preliminary remarks are in order. Let $\mathcal{L}_{\mathrm{PA}}$ be the usual language of PA: relations $+, \cdot, S$, and $<$; and constants 0 and 1. We say that a sentence is B_n if it is equivalent to a Boolean combination of Σ_n sentences. All models here are countable. We will measure the complexity of a structure by the Turing degree of its atomic diagram. For simplicity, we will identify formulas with their Gödel numbers.

In [1], we showed the following independence result.

Theorem 1.1. *For every n, given a computably enumerable (c.e.) index for a consistent set of axioms P extending PA, we can effectively find a Π_n sentence φ such that:*

(a) *for any set Ψ of B_{n-1} sentences consistent with P, $P + \Psi + \varphi$ is consistent; and*
(b) *for any set Λ of Σ_n sentences consistent with P, $P + \Lambda + \neg\varphi$ is consistent.*

Using Gödel's fixed point lemma, we may state φ as follows:

$\forall p \forall \overline{u}[(p$ is a proof of me from P + true B_{n-1} sentences + one Σ_n sentence $\exists \overline{x}\chi(\overline{x})$ such that $\mathrm{Sat}_{\Pi_{n-1}}(\chi, \overline{u})) \to \exists q < p \exists \overline{u}' \leq \overline{u}(q$ is a

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proof of my negation from P + true B_{n-1} sentences + one true Σ_n sentence $\exists\overline{x}\chi'(\overline{x})$ such that $\mathrm{Sat}_{\Pi_{n-1}}(\chi', \overline{u}'))]$

We also showed in [1] the following extension of Theorem 1.1.

THEOREM 1.2. *For every n and every $k \geq 1$, given a c.e. index for a consistent set of axioms P extending PA, we can effectively find Π_n sentences $\{\varphi_i : 1 \leq i < k\}$ such that:*

(a) *for any set Ψ of B_{n-1} sentences consistent with P, $P + \Psi + \varphi_1 + \cdots + \varphi_k$ is consistent; and*
(b) *for any set Λ of Σ_n sentences consistent with P and any $1 \leq i \leq k$, if $P + \Lambda + \varphi_{i+1} + \cdots + \varphi_k$ is consistent, then so is $P + \Lambda + \neg\varphi_i + \varphi_{i+1} + \cdots + \varphi_k$.*

To obtain the sequence described in Theorem 1.2, we start by letting φ_k be the φ of Theorem 1.1. We continue by adding φ_k to P and applying Theorem 1.1 to get the next sentence φ_{k-1}. Repeating this $k-1$ many times, we produce the desired sequence of sentences.

We were interested in these two results primarily because we sought the following two results (cf. [1] for precisely why).

COROLLARY 1.3. *For all n, there exist T, T', consistent, complete extensions of PA, such that*

$$T \cap \Sigma_n \subset T' \cap \Sigma_n$$

with $\varphi \in T$, and $\neg\varphi \in T'$, for φ as in Theorem 1.1.

COROLLARY 1.4. *For all n and all $k \geq 1$, there exists a sequence $\{T^i\}_{i \leq k}$ of consistent, complete extensions of PA such that*

$$T^0 \cap \Sigma_n \subset T^1 \cap \Sigma_n \subset \ldots \subset T^k \cap \Sigma_n,$$

and $\varphi_i \in T^j$ iff $j < i \leq k$, for $\{\varphi_i\}_{1 \leq i < k}$ as in Theorem 1.2.

Leo Harrington and Julia Knight, independently, asked if Theorem 1.2 could be extended to yield an infinite sequence of sentences with the described feature. In Theorem 1.2, each sentence φ_i referenced the sentences φ_j, for $i < j \leq k$. The problem with proceeding as we did for Theorem 1.2 is that there is no 'last' sentence with which to start. Therefore, we cannot obtain an infinite sequence by iterating Theorem 1.1 a finite number of times as we did for Theorem 1.2. We must proceed differently.

By dropping the requirement that the sequence of sentences be computable, though, we can answer Harrington and Knight's question affirmatively. That is, there is a sequence of sentences with the desired features, as the following result makes precise.

THEOREM 1.5. *Let P be a consistent, c.e. set of sentences including the axioms of PA. For every n, there is a sequence of Π_n sentences $\{\varphi_i\}_{i\in\omega}$ such that the following hold:*

(a) *For any set Ψ of B_{n-1} sentences such that $P+\Psi$ is consistent, $P+\Psi+\{\varphi_i : i\in\omega\}$ is also consistent.*
(b) *For any set Λ of Σ_n sentences and any k such that $P+\Lambda+\{\varphi_i : i>k\}$ is consistent, $P+\Lambda+\{\neg\varphi_k\}+\{\varphi_i : i>k\}$ is also consistent.*

We again obtain a corollary for completions of PA.

COROLLARY 1.6. *For all n, there exists a sequence $\{T^i\}_{i\in\omega}$ of consistent, complete extensions of PA such that for $i<j$,*

$$T^i \cap \Sigma_n \subset T^j \cap \Sigma_n,$$

and $\varphi_i \in T^i, \neg\varphi_i \in T^j$.

In fact, we obtain the following stronger result, which entails Theorem 1.5. This is our main result in this paper.

THEOREM 1.7. *For all $n\geq 1$, there exists a computable function F with domain H, where $(H,<_H)$ is a computable linear ordering of order type $\omega_1^{CK}(1+\eta)$, such that*

(a) *$F(a)$ is a Π_n sentence in $\mathcal{L}_{\mathrm{PA}}$, for all $a\in H$;*
(b) *for any set Ψ of B_{n-1} sentences consistent with PA, $\mathrm{PA}+\Psi+\{F(b) : b\in H\}$ is consistent; and*
(c) *for all $a\in H$ and any set Λ of Σ_n sentences consistent with PA, if $\mathrm{PA}+\Lambda+\{F(b) : b<_H a\}$ is consistent, then $\mathrm{PA}+\Lambda+\{F(b) : b<_H a\}+\{\neg F(a)\}$ is also consistent.*

Theorem 1.7 also yields the following corollary for completions of PA, stronger than Corollary 1.6.

COROLLARY 1.8. *For all n, there is a linear ordering $(H,<_H)$ of order type $\omega_1^{CK}(1+\eta)$ and a sequence $\{T^i\}_{i\in H}$ of consistent, complete extensions of PA such that for all $a<_H b$,*

$$T^a \cap \Sigma_n \subset T^b \cap \Sigma_n,$$

and $\varphi_a \in T^a, \neg\varphi_a \in T^b$.

In [3], P. D'Aquino and J. Knight have recently used Theorem 1.7 to solve a problem concerning weak fragments of arithmetic. To present their result, we need a few preliminaries. The fragment $I\Delta_0$ of PA consists of the axioms of PA, with the induction schema restricted to B_0 formulas. Similarly, the fragments $I\Sigma_n$ for each n

consist the axioms of PA, with the induction schema restricted to Σ_n formulas.

We say that a set A is *representable* in T if there is a formula $\varphi(x)$ such that

$$n \in A \Rightarrow T \vdash \varphi(S^n(0)) \text{ and } n \notin A \Rightarrow T \vdash \neg\varphi(S^n(0)).$$

We denote the family of all such sets by $Rep(T)$.

S. Feferman observed in [4] that for a completion T of PA and a nonstandard $\mathcal{A} \models T$, the sets coded in T are all coded in $\mathcal{A}$. More precisely, he observed that $Rep(T) \subseteq \mathcal{SS}(\mathcal{A})$. D'Aquino and Knight have used Theorem 1.7 to show that this does not hold for any of the fragments $I\Delta_0$, $I\Sigma_n$, for all n. They have shown that there is a completion T of $I\Delta_0$, and a nonstandard model $\mathcal{A} \models T$, such that $T \cap \Sigma_1 \in Rep(T) - \mathcal{SS}(\mathcal{A})$. They also showed that for each $n \geq 1$, there is a completion T of $I\Sigma_n$, and a model $\mathcal{A} \models T$, such that $T \cap \Sigma_{n+1} \in Rep(T) - \mathcal{SS}(\mathcal{A})$. In fact, this holds even while taking $T \cap B_n$ to be the B_n part of an arbitrary completion of PA.

In Section 3, we use the Barwise–Kreisel Compactness Theorem to prove the main result, Theorem 1.7. The next section gives the necessary background for that proof.

§2. Background for the upcoming proofs. We assume that the reader has some background in computable ordinals and the hyperarithmetical hierarchy. Our main references here are [6], Chapter 5, and [2], especially Chapters 4 through 8. We indicate below the key notions and results that we need for what follows.

2.1. Satisfaction. The sentence φ given by in Theorem 1.1 uses the predicates $\mathrm{Sat}_{\Sigma_n}(x,y)$ and $\mathrm{Sat}_{\Pi_n}(x,y)$. These predicates define satisfaction in PA for Σ_n and Π_n formulas, respectively. Thus $\mathrm{Sat}_{\Pi_n}(x,y)$ expresses that the Π_n formula $\varphi(y)$ coded by x is satisfied by the element y. For each n, $\mathrm{Sat}_{\Sigma_n}(x,y)$ and $\mathrm{Sat}_{\Pi_n}(x,y)$ are Σ_n and Π_n sentences, respectively. We refer the reader to Kaye's book on models of PA for more on defining satisfaction [5].

2.2. Computable ordinals. As is well known, the computable ordinals are the same as the ordinals for which we can assign a "notation". To say that a computable ordinal α has notation a, we say $|a| = \alpha$. Let O denote Kleene's system of ordinal notations. Let $<_o$ be Kleene's partial ordering on O. We denote the least noncomputable ordinal by ω_1^{CK}. Let η denote the order type of the rationals.

We use the following result frequently in what follows.

PROPOSITION 2.1. *For $a \in O$, $\{b : b <_o a\}$ has order type $|a|$ under $<_o$ and is c.e., uniformly in a.*

2.3. The analytical hierarchy. A relation $S(\overline{x})$ is called Π^1_1 if

$$S(\overline{x}) \Leftrightarrow \forall f \exists t R(\overline{x}, t, f \upharpoonright t),$$

where the quantifier $\forall f$ ranges over function variables, the quantifier $\exists t$ ranges over individual variables, and R is a computable relation. Similarly, a relation $S(\overline{x})$ is called Σ^1_1 if

$$S(\overline{x}) \Leftrightarrow \exists f \forall t R(\overline{x}, t, f \upharpoonright t),$$

where R is a computable relation. If a relation is both Π^1_1 and Σ^1_1, we say that it is Δ^1_1. The following classical result of Kleene says that the Δ^1_1 sets are exactly the hyperarithmetical sets:

THEOREM 2.2 (Kleene). *For any set S, the following are equivalent:*

(a) *S is Δ^1_1,*
(b) *S is hyperarithmetical.*

2.4. Computable infinitary sentences. In Section 3, we will describe a set of axioms that we will show has a computable model. These axioms, however, will not be finitary. We will take as our language $\mathcal{L}_{\omega_1\omega}$. The difference between sentences in $\mathcal{L}_{\omega_1\omega}$ and sentences in an elementary first-order language is that these sentences may have countable conjunctions and disjunctions. For example, the following sentence is expressible in $\mathcal{L}_{\omega_1\omega}$, where $\mathcal{L}$ includes a function symbol S:

$$\forall x \bigvee_n x = S^{(n)}(0).$$

We will be especially concerned in this paper with infinitary sentences whose infinite conjunctions and disjunctions range over c.e. sets. The example above clearly has this feature. The members of this restricted class of infinitary sentences are called *computable infinitary sentences*. We will see some of the special features of this class in the remainder of this section.

We use the following facts in what follows, often without saying.

FACT 2.3. *For any hyperarithmetic structure $\mathcal{A}$, the set of all computable infinitary sentences true in $\mathcal{A}$ is Π^1_1.*

Fact 2.4. *For any two hyperarithmetic structures $\mathcal{A}, \mathcal{B}$, if $\mathcal{B}$ satisfies all of the computable infinitary sentences true in $\mathcal{A}$, then $\mathcal{B} \simeq \mathcal{A}$.*

Fact 2.5. *For any $\alpha < \omega_1^{CK}$, there exists $\beta_0 < \omega_1^{CK}$ such that for all countable $\gamma, \beta \geq \beta_0$, $\omega^\gamma(1+\eta) \leq_\alpha \omega^\beta$, and $\omega^\beta \leq_\alpha \omega^\gamma(1+\eta)$.*

The relation $\leq_\alpha$ in Fact 2.5 is the back-and-forth relation for linear orderings (cf. [2], Section 15.3.3 for more details). It follows from Fact 2.5 that the linear ordering $\omega_1^{CK}(1+\eta)$ satisfies the same computable infinitary sentences as ω_1^{CK}. It also follows that for any $\alpha < \omega_1^{CK}$, there is some $\beta < \omega_1^{CK}$ such that the computable Π_α sentences true in $\omega_1^{CK}(1+\eta)$ are true in β.

2.5. The Barwise–Kreisel Compactness Theorem. Though the usual compactness theorem fails when we consider theories in $\mathcal{L}_{\omega_1\omega}$, there is a modification for computable infinitary sentences that works, considering Δ_1^1 subsets instead of finite ones.

Theorem 2.6 (Barwise–Kreisel Compactness). *Let Γ be a Π_1^1 set of computable infinitary sentences such that every Δ_1^1 subset of Γ has a model. Then Γ has a model.*

In fact, we will use a corollary of this result.

Corollary 2.7 (Barwise–Kreisel Compactness). *Let Γ be a Π_1^1 set of computable infinitary sentences such that every Δ_1^1 subset of Γ has a computable model. Then Γ has a computable model.*

Using Corollary 2.7, we may prove the following result, due to Harrison.

Theorem 2.8 (Harrison). *There is a computable linear ordering $(H^*, <_{H^*})$ of order type $\omega_1^{CK}(1+\eta)$.*

We also have the following.

Proposition 2.9. *Any Δ_1^1 set of computable infinitary sentences true in Harrison's ordering $(H^*, <_{H^*})$ is also true in ω^α, for a sufficiently large computable ordinal α.*

Proposition 2.9 follows from Fact 2.5.

§3. Proof of Theorem 1.7. Here is our strategy for this section. We will begin by giving a Π_1^1 set Γ of computable infinitary sentences describing a structure

$$\langle H \cup \omega, <_H, F, H, \omega, +, \cdot, 0, S \rangle$$

which, if computable, has the desired properties for Theorem 1.7. Next, we show that every Δ^1_1 subset of Γ has a model and apply the Barwise–Kreisel Compactness Theorem to show that Γ has a model. This suffices for proving Theorem 1.7. We may then obtain Theorem 1.5 by using the non-well-founded part of this model to show that the desired sentences exist.

3.1. Specifying a set Γ of computable infinitary sentences. The set Γ describes a structure

$$\langle H \cup \omega, <_H, F, H, \omega, +, \cdot, 0, S\rangle.$$

The sentences in Γ express the following five properties.

(1) $\langle \omega, +, \cdot, 0, S\rangle$ is (a copy of) the standard model of arithmetic, in which we can talk about sentences, proofs, etc. We use the standard names for easy recognition.

(2_σ) $(H, <_H) \models \sigma$ for all computable infinitary sentences σ true in $(H^*, <_{H^*})$, the Harrison ordering of Theorem 2.8.

(3) $F : H \to \omega$ is a function taking elements in H to elements of ω representing sentences in $\mathcal{L}_{\text{PA}}$.

(4_ψ) For a B_{n-1} sentence ψ in $\mathcal{L}_{\text{PA}}$, if there is no proof of a contradiction from PA + ψ, then there is no proof of a contradiction from PA + ψ + $Range(F)$.

(5_λ) for a Σ_n sentence λ in $\mathcal{L}_{\text{PA}}$, for all a, if $\text{PA} + \lambda + \{F(b) : b <_H a\}$ is consistent, then $\text{PA} + \lambda + \{\neg F(a)\} + \{F(b) : b <_H a\}$ is also consistent.

We indicate how to express a few of these properties using computable infinitary sentences. For more information, see [2].

To express property (1) in a computable infinitary sentence φ, take φ to be the conjunction of the axioms of PA plus the sentence

$$\forall x[x \in \omega \to \bigvee\!\!\!\bigvee_n x = S^{(n)}(0)],$$

following Section 6.2, Example 1, of [2]. To express property (5_λ) for each Σ_n sentence λ in $\mathcal{L}_{\text{PA}}$, we use a sentence saying that for all $a \in H$, if there is no proof $p \in \omega$ of $0 = 1$ from axioms PA, λ, and sentences in $\{F(b) : b <_H a\}$, then there is no proof $p \in \omega$ of $0 = 1$ from axioms PA, λ, sentences in $\{F(b) : b <_H a\}$, and $\neg F(a)$.

Next, we argue that Γ is Π^1_1. The set Γ consists of a set Γ_0 of sentences describing the Harrison ordering, and further sentences, expressing properties (1) and (3)-(5) above, forming a set Γ_1. In fact, Γ_1 is computable, so it suffices to argue that Γ_0 is Π^1_1. Since the Harrison ordering is hyperarithmetical (in fact, computable), it follows by Fact 2.3 that Γ_0 is Π^1_1.

3.2. Considering Δ^1_1 subsets of Γ. As we just pointed out, Γ consists of a Π^1_1 set Γ_0 of sentences describing the Harrison ordering, and further sentences forming a set Γ_1. Let Γ' be a Δ^1_1 subset of Γ. By Proposition 2.9, $\Gamma'_0 = \Gamma' \cap \Gamma_0$ is true in an ordering of order type $\alpha = \omega^\beta$ for some computable ordinal β. We must find a computable model of both Γ'_0 and Γ'_1, however.

To do this, we begin by giving a model of the form

$$\langle H_a \cup \omega, <_{H_a}, F_a, H_a, \omega, +, \cdot, 0, S\rangle,$$

where a is a notation for α, $H_a = \{b : b <_o a\}$, a c.e. set, and F_a is a partial computable function defined on H_a. We will later replace this structure by an isomorphic one that is computable.

Toward F_a, we define a partial computable function f with domain O, by computable transfinite recursion on ordinal notations. This function is described by the following lemma.

LEMMA 3.1. *For every n, there is a partial computable function $f : O \to \omega$ with the following properties for every $a \in O$:*

(a) *$f(a)$ is a sentence in $\mathcal{L}_{\mathrm{PA}}$;*
(b) *for any set Ψ of B_{n-1} sentences in $\mathcal{L}_{\mathrm{PA}}$, if $\mathrm{PA}+\Psi$ is consistent, then $\mathrm{PA}+\Psi+\{f(b) : b <_o a\}$ is also consistent; and*
(c) *for any set Λ of Σ_n sentences in $\mathcal{L}_{\mathrm{PA}}$, if $\mathrm{PA}+\Lambda+\{f(b) : b <_o a\}$ is consistent, then $\mathrm{PA}+\Lambda+\{\neg f(a)\}+\{f(b) : b <_o a\}$ is also consistent.*

Let F_a be the restriction of f to H_a. Lemma 3.1 entails that F_a satisfies properties (3), (4), and (5), when these properties are restricted to H_a as follows.

(3) $F : H \to \omega$ is a function taking elements in H to elements of ω representing sentences in $\mathcal{L}_{\mathrm{PA}}$.
(4_ψ) For a B_{n-1} sentence ψ in $\mathcal{L}_{\mathrm{PA}}$, if there is no proof of a contradiction from $\mathrm{PA}+\psi$, then there is no proof of a contradiction from $\mathrm{PA}+\psi+\mathit{Range}(F)$.
(5_λ) for a Σ_n sentence λ in $\mathcal{L}_{\mathrm{PA}}$, for all a, if $\mathrm{PA}+\lambda+\{F(b) : b <_H a\}$ is consistent, then $\mathrm{PA}+\lambda+\{\neg F(a)\}+\{F(b) : b <_H a\}$ is also consistent.

We may use padding to obtain an isomorphic copy of our structure

$$(H_a, <_{H_a}, F_a) \cong_g (H^*_a, <_{H^*_a}, F^*_a),$$

where g is partial computable and H_a^* is computable. The new ordering $<_{H_a^*}$ and the function F_a^* will also be computable, so we get a computable model of Γ'.

All that remains for us to do in this section is to prove Lemma 3.1. That gap is filled by the following argument.

Proof of Lemma 3.1. Fix n. For $a \in O$, here is how we define $f(a)$:

Case 1: Suppose $a = 1$.

Let $f(1)$ be the sentence φ from Theorem 1.1, with $P = \mathrm{PA}$.

Case 2: Suppose $a = 2^d$.

By our inductive hypothesis we have an index for $\{f(b) : b \leq_o d\}$. Let $f(a)$ be the sentence φ_a obtained using Theorem 1.1, with $P = \mathrm{PA} + \{f(b) : b \leq_o d\}$. We can also obtain an index for $\{f(b) : b \leq_o a\}$.

Case 3: Suppose $a = 3 \cdot 5^e$.

Then a is the least upper bound of the sequence $\{\varphi_e(n)\}_{n \in \omega}$. By our inductive hypothesis we have an index for $\{f(b) : b <_o k\}$ for all k such that $\varphi_e(n) = k$ for some n. But

$$\{f(b) : b <_o a\} = \bigcup_k \{f(b) : b <_o k\}.$$

Since we have a computable sequence of indices for the c.e. sets $\{f(b) : b <_o k\}$, and we have a c.e. index for the sequence, their union $\{f(b) : b <_o a\}$ is also c.e., and hence we have a c.e. index for $\{f(b) : b <_o a\}$. We may then let $f(3 \cdot 5^e)$ be the sentence φ_a obtained using Theorem 1.1, with $P = \mathrm{PA} + \{f(b) : b <_o a\}$. We can also obtain an index for $\{f(b) : b \leq_o a\}$.

This completes our definition of f by computable transfinite recursion on ordinal notations. It is clear from our definition that (a) holds. It follows from Theorem 1.1 and transfinite induction on ordinal notations that (b) and (c) hold. $\dashv$

Having shown that every Δ_1^1 subset of Γ has a computable model, we may apply Corollary 2.7 and obtain a computable model $\mathcal{H}$ of Γ. This completes the proof of Theorem 1.7.

To prove Theorem 1.5, we must identify the sentences $\{\varphi_i\}_{i \in \omega}$. Here is how to obtain such a sequence of sentences. Start by taking some $b_0 \in H$ from the non-well-founded part, i.e., such that b_0 is not in any of the initial segments of order type α for α a computable ordinal. There must be such a b_0, for if there were not then $\mathcal{H}$ would have order type ω_1^{CK}. But this is impossible, since $\mathcal{H}$ is computable.

There can be no first element in the non-well-founded part either, because if there were then there would be a computable initial segment containing all the elements less than this first element, again impossible. We thus obtain an infinitely decreasing sequence $\{b_i\}_{i\in\omega}$ of elements of H such that no b_i is in an initial segment of order type α for α a computable ordinal. We may then give the sequence of sentences satisfying Theorem 1.5. For all $i \in \omega$, let $\varphi_i = F(b_i)$.

§4. Another application. This method also enables us to prove the following result.

THEOREM 4.1. *There exists a computable function F with domain H, where $(H, <_H)$ is a computable linear ordering of order type $\omega_1^{CK}(1+\eta)$, such that for all $a \in H$,*
(a) *$F(a)$ is a Π_1 sentence in $\mathcal{L}_{\mathrm{PA}}$;*
(b) *$\mathrm{PA} + F(a)$ is consistent; and*
(c) *for any $b <_H a$, $\mathrm{PA} \vdash F(a) \to Con(\mathrm{PA} + F(b))$.*

However, the following, stronger result was proved by A. Wilkie (cf. [7]).

THEOREM 4.2 (Wilkie). *There is a computable sequence of Π_1 sentences $\{\varphi_a\}_{a\in H}$, where $(H, <_H)$ is a computable linear ordering of order type η, such that for all $a \in H$, $\mathrm{PA} + \varphi_a$ is consistent, and $\mathrm{PA} + \varphi_a \vdash Con(\mathrm{PA} + \varphi_b)$ for $a <_H b$.*

Prior to Wilkie, and independently of each other, H. Friedman, C. Smoryński, and R. Solovay gave a computable sequence of order type ω, rather than η.

THEOREM 4.3 (Friedman, Smoryński, Solovay). *There is a computable sequence of Π_1 sentences $\{\varphi_i\}_{i\in\omega}$ such that for all $i \in \omega$, φ_i is consistent with PA and $\mathrm{PA} + \varphi_i \vdash Con(\mathrm{PA} + \varphi_{i+1})$.*

Given Theorem 4.2, we could substitute into it any desired computable order type, so Theorem 4.2 implies both Theorem 4.1 and Theorem 4.3.

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