SOLOVAY'S THEOREM CANNOT BE SIMPLIFIED

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Abstract. In this paper we consider three potential simplifications to a result of Solovay's concerning the Turing degrees of nonstandard models of arbitrary completions of first-order Peano Arithmetic (PA). Solovay characterized the degrees of nonstandard models of completions T of PA, showing that they are the degrees of sets X such that there is an enumeration $R \leq_T X$ of an "appropriate" Scott set and there is a family of functions $(t_n)_{n\in\omega}$, $\Delta^0_n(X)$ uniformly in n, such that $\lim_{s\to\infty} t_n(s)$ is an index for $T \cap \Sigma_n$ and for all s, $t_n(s)$ is an index for a subset of $T \cap \Sigma_n$. The simplifications under consideration are attempts to restrict the families of functions $(t_n)_{n\in\omega}$ that appear in Solovay's result, known henceforth as *Solovay families*. We show that none of these potential simplifications may be made, by proceeding as follows. First, we construct a nonstandard model $\mathcal{A}$ of PA such that there is no Solovay family $(t_n)_{n\in\omega}$ for $\mathrm{Th}(\mathcal{A})$ relative to $\mathcal{A}$ in which all the functions t_n are constant. Second, for each k we construct a nonstandard model $\mathcal{A}$ of PA such that there is no Solovay family $(t_n)_{n\in\omega}$ for $\mathrm{Th}(\mathcal{A})$ relative to $\mathcal{A}$ in which all the functions t_n change values at most k many times. Third, we construct a nonstandard model $\mathcal{A}$ of PA such that there is no Solovay family $(t_n)_{n\in\omega}$ for $\mathrm{Th}(\mathcal{A})$ relative to $\mathcal{A}$ with a computable function f such that for all n, $t_n(s)$ changes values at most $f(n)$ times. Our constructions answer three questions asked by Julia Knight [7]. Our solutions make use of several consistency results that seem to be of independent interest.

1. Introduction

We are concerned here with three potential simplifications to a result of Robert Solovay characterizing the degrees of nonstandard models of arbitrary completions of PA. In this introductory section we will begin by reviewing some notions related to Solovay's result. We will then discuss Solovay's work and some related issues.

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Lastly, we will discuss the proposed simplifications to Solovay's result that will be our concern in the remainder of the paper.

We use $P(\omega)$ to denote the class of all subsets of ω. Let $\mathcal{L}_{\mathrm{PA}}$ be the usual language of PA: relations $+, \cdot, S$, and $<$; and constants 0 and 1. We abbreviate True Arithmetic, the theory of the standard model of PA, by the initials TA. We use $S^n(0)$ to denote the numeral for n. For simplicity, we identify sentences with their Gödel numbers.

We continue by introducing the notion of a *Scott set*. Dana Scott introduced the notion in exploring what sets are coded in completions and models of PA [11].

Definition 1.1. A *Scott set* is a nonempty family of sets $\mathcal{S} \subseteq P(\omega)$ such that

(1) if $X \in \mathcal{S}$ and $Y \leq_T X$, then $Y \in \mathcal{S}$,
(2) if $X, Y \in \mathcal{S}$, then $X \oplus Y \in \mathcal{S}$,
(3) if $T \subseteq 2^{<\omega}$ is an infinite tree in $\mathcal{S}$, then T has a path in $\mathcal{S}$. Equivalently, if A is a consistent set of sentences in $\mathcal{S}$, then a complete extension of A is in $\mathcal{S}$.

As an example, the family of arithmetical sets forms a Scott set. Additionally, Scott sets are the ω-models of the axiom system WKL_0 studied in reverse mathematics (where the model is identified with the power set part of the structure, as in [12]).

Scott's interest in these families originated in his investigation of what sets are "represented" in a completion of PA. A set A is *representable* in T if there is a formula $\varphi(x)$ such that

$$n \in A \Rightarrow T \vdash \varphi(S^n(0)) \text{ and } n \notin A \Rightarrow T \vdash \neg\varphi(S^n(0)).$$

We denote the family of all such sets by *Rep*(T). Scott showed the following:

Theorem 1.2 (Scott)**.** *The following are equivalent:*

(1) $\mathcal{S}$ *is a countable Scott set*
(2) *there is a completion* T *of PA such that Rep*$(T) = \mathcal{S}$.

A related notion is that of an "appropriate" Scott set. We say that a Scott set $\mathcal{S}$ is *appropriate* for a completion T of PA iff $T_n = T \cap \Sigma_n \in \mathcal{S}$, for all n.

Scott also investigated what sets are coded in models of PA. Let

$$d_a = \{n \in \omega : \mathcal{A} \models p_n | a\},$$

where $(p_n)_{n\in\omega}$ is the sequence of primes. He showed the following:

Theorem 1.3 (Scott)**.** *Let $\mathcal{A}$ be a nonstandard model of PA, and let*

$$\mathcal{SS}(\mathcal{A}) = \{d_a : a \in \mathcal{A}\}.$$

Then $\mathcal{SS}(\mathcal{A})$ is a Scott set.

We call $\mathcal{SS}(\mathcal{A})$ *the Scott set of* $\mathcal{A}$. Feferman observed the following fact about Scott sets of models [3]:

Theorem 1.4 (Feferman)**.** *If $Th(\mathcal{A}) = T$, then $T \cap \Sigma_n \in \mathcal{SS}(\mathcal{A})$, for all n. Hence, $Rep(T) \subseteq \mathcal{SS}(\mathcal{A})$.*

Thus $\mathcal{SS}(\mathcal{A})$ is appropriate for T, for $\mathcal{A}$ a nonstandard model of T.

A notion we will use in connection with Scott sets is that of an "enumeration".

Definition 1.5. An *enumeration* of a set $\mathcal{S} \subseteq P(\omega)$ is a binary relation R such that $\mathcal{S} = \{R_n : n \in \omega\}$, where

$$R_n = \{k : (n, k) \in R\}.$$

An *R-index* for X is some $k \in \omega$ such that $R_k = X$.

For a nonstandard model $\mathcal{A}$ of PA, $R = \{(a, n) : \mathcal{A} \models p_n | a\}$ is called the *canonical enumeration* of $\mathcal{SS}(\mathcal{A})$.

Solovay defined the notion of an "effective enumeration".

Definition 1.6. For a countable Scott set $\mathcal{S}$, an *effective enumeration* is an enumeration R, with associated functions f, g, and h witnessing that $\mathcal{S}$ is a Scott set. These functions have the following properties:

(1) if $\varphi_e^{R_i} = \chi_X$, then $f(i, e)$ is an R-index for X,
(2) $g(i, j)$ is an R-index for $R_i \oplus R_j$,

(3) if R_i is an infinite tree $T \subseteq 2^{<\omega}$, then $h(i)$ is an R-index for a set X such that χ_X is a path through T.

We say that an effective enumeration is computable in a set X if the enumeration and the three functions are all computable in X. Effective enumerations are available to us in light of the following result due to David Marker [8]:

Theorem 1.7 (Marker)**.** *Let $\mathcal{S}$ be a countable Scott set. If $\mathcal{S}$ has an enumeration computable in X, then it also has an effective enumeration computable in X.*

Let's now turn our attention to Solovay's results. Carl Jockusch had asked for a characterization of the degrees of models of True Arithmetic (TA), the theory of the standard model of PA; he also asked what could be said about the degrees of models of other specific completions of PA. In [13], Solovay gave a characterization of the degrees of nonstandard models of TA, but his characterization used effective enumerations. Combining Solovay's result with Marker's, we have the following simpler characterization:

Theorem 1.8 (Solovay / Marker)**.** *The degrees of nonstandard models of TA are the degrees of enumerations of Scott sets containing the arithmetical sets.*

Solovay next turned his attention to arbitrary completions of PA. Some years after characterizing the degrees of nonstandard models of TA, Solovay characterized the degrees of nonstandard models of arbitrary completions T of PA. The result is more difficult to state than the result for TA. To see why, we highlight an important difference between TA and arbitrary completions of PA. It is well-known that for all n, the fragment $\mathrm{TA}_n := \mathrm{TA} \cap \Sigma_n \equiv_T \emptyset^{(n)}$. By Feferman's result (Theorem 1.4), $\mathrm{TA} \cap \Sigma_n \in \mathcal{SS}(\mathcal{A})$, for $\mathcal{A}$ a nonstandard model of TA. It follows that $\mathcal{SS}(\mathcal{A})$ contains the arithmetical sets. Therefore, if R is an enumeration of $\mathcal{SS}(\mathcal{A})$, then computably in $\mathcal{A}''$ we can compute a sequence $(i_n)_{n\in\omega}$ of indices such that $R_{i_k} = \mathrm{TA}_k$ for each k.

However, for an arbitrary completion T of PA we may not be able to compute indices for all fragments of T in just two jumps of a nonstandard model, as results in Section 4 of [1] illustrate. The difference is that the fragments of T may not correspond with the arithmetical sets in any clear way. In fact, for some completions

T we cannot compute the R-indices of fragments of T from any finite number of jumps of a model. This is why Solovay's characterization for arbitrary completions T of PA must contain an extra condition giving a family of functions t_n, $\Delta^0_n(\mathcal{A})$, such that in the limit t_n gives an R-index for T_n. The extra condition requires not just the oracle $\Delta^0_3(\mathcal{A})$ as in the TA case, but rather the sequence of oracles $(\Delta^0_n(\mathcal{A}))_{n\in\omega}$.

Let's pay special attention to the features of this family of functions. A *Solovay family* for T relative to X is a family of functions t_n, $\Delta^0_n(X)$, uniformly in n, s.t. $\lim_{s\to\infty} t_n(s)$ is an index for $T \cap \Sigma_n$, and for all s, $t_n(s)$ is an index for a subset of $T \cap \Sigma_n$ (following the terminology of [7]). We may take these indices to be either R-indices in some enumeration R of a Scott set appropriate for T, or indices for T as a set computable in a nonstandard model $\mathcal{A}$ of T.

As our previous discussion made clear, our main interest is when $X = \mathcal{A}$ and $T = Th(\mathcal{A})$. By Feferman's Theorem 1.4, $T \cap \Sigma_n \in \mathcal{SS}(\mathcal{A})$. Thus all of the theory's fragments are coded in the Scott set — only we have no indication where they are. The power of Solovay families is that they allow us to locate these coded fragments by computing their indices (relative to oracles $\Delta^0_n(\mathcal{A})$). The striking fact that such families exist for all completions of PA is due to Solovay in the following result:

Theorem 1.9 (Solovay)**.** *For T a completion of PA, $\mathcal{A} \models T$, there is a Solovay family for T relative to $\mathcal{A}$.*

Theorem 1.9 gives half of a characterization for arbitrary completions of PA; the other half is filled in by the following theorem, which we call *Solovay's Theorem*:

Theorem 1.10 (Solovay)**.** *Suppose T is a completion of PA. The degrees of nonstandard models of T are the degrees of sets X such that:*

(a) *There is an enumeration $R \leq_T X$ of a Scott set $\mathcal{S}$ appropriate for T; and*
(b) *There are functions t_n for $n \geq 1, \Delta^0_n(X)$ uniformly in n, such that $\lim_{s\to\infty} t_n(s)$ is an index for T_n and for all s, $t_n(s)$ is an R-index for a subset of T_n.*

Solovay did not publish these results we are attributing to him. In [6], Julia Knight has given proofs of Theorems 1.8 and 1.10. The proofs she gives are not Solovay's proofs; in each case, though, she found a proof after learning from Solovay that the result is true.

For simplicity, we have so far identified nonstandard models of PA with their open diagrams. Recall that the open diagram of a structure $\mathcal{A}$, denoted $D(\mathcal{A})$, is the collection of open sentences, with constants from $\mathcal{A}$, that are true in $\mathcal{A}$. Solovay's results characterize the degrees of open diagrams of nonstandard models of completions of PA. We might also investigate the degrees of diagrams of higher arithmetical complexity. A relevant notion here is that of the "m-diagram" of a model. The *m-diagram* of $\mathcal{A}$, denoted $D_m(\mathcal{A})$, is the collection of B_m sentences, with constants from $\mathcal{A}$, that are true in $\mathcal{A}$. A B_m formula is a boolean combination of Σ_m formulas.

Investigations of extensions to Solovay's results to m-diagrams can be found in [1]. The following results are the central ones there:

Theorem 1.11 (Arana)**.** *For any $m \geq 0$, the degrees of m-diagrams of nonstandard models of TA are the degrees of enumerations of Scott sets containing the arithmetical sets.*

This result yields the following corollary:

Corollary 1.12. *The degrees of m-diagrams of nonstandard models $\mathcal{A}$ of TA are the same for all $m \geq 0$.*

Solovay's Theorem 1.10 for arbitrary completions of PA may also be extended to m-diagrams as follows:

Theorem 1.13 (Arana)**.** *Suppose T is a completion of PA. For any $m \geq 0$, the degrees of m-diagrams of nonstandard models of T are the degrees of sets X such that:*

(a) *There is an enumeration $R \leq_T X$ of a Scott set $\mathcal{S}$ appropriate for T; and*
(b) *There are functions t_{m+n} for $n \geq 1, \Delta^0_n(X)$ uniformly in n, such that $\lim_{s\to\infty} t_{m+n}(s)$ is an index for T_{m+n} and for all s, $t_{m+n}(s)$ is an R-index for a subset of T_{m+n}.*

Solovay families have surprising features for completions of PA. Not only do these functions approximate fragments of theories, but their approximations never give false information. If the functions are not constant, then their first approximation of the fragment will be a subset of the fragment. After they change values finitely many times, they will yield the R-index for the fragment. However, in the examples using Solovay's results known until now, the functions in the relevant Solovay family have been constant. For instance, we may use Solovay's Theorem to prove Harrington's result that there exists a nonstandard model $\mathcal{A} \models PA$ such that $\mathcal{A} \leq_T 0'$ but $Th(\mathcal{A})$ is not arithmetical [4]. To get such a model $\mathcal{A}$, we take an enumeration $R \leq_T \emptyset'$, then specify a completion T and a Solovay family $(t_n)_{n\in\omega}$ with some special features. Lastly, we apply Solovay's Theorem to get $\mathcal{A} \leq_T R$. The Solovay family we use consists entirely of constant functions, so that for all n and s, $t_n(s) = i_n$ and $R_{i_n} = T_n$.

Such considerations might lead us to try to simplify Solovay's Theorem, for instance by reducing condition (b) to require only a family of constant functions yielding indices for the fragments of the theory. Knight has pointed out three particular simplifications that one might try to make to Solovay's Theorem. We might take the Solovay family to consist exclusively of constant functions; or functions that change on the index for T_n at most once; or functions that change on the index for T_n at most $f(n)$ times for some computable function f. In [7], she stated three problems that rule out these simplifications of Solovay's Theorem, where they appear as Problems 2, 3, and 4. Here are the three problems:

(1) Construct a nonstandard model $\mathcal{A}$ of PA such that there is no Solovay family $(t_n)_{n\geq 1}$ for Th($\mathcal{A}$) relative to $\mathcal{A}$ in which all the functions t_n are constant. That is, find $\mathcal{A}$ such that there is no uniformly effective procedure in n using $\Delta^0_n(\mathcal{A})$ to find an index for $Th(\mathcal{A}) \cap \Sigma_n$ as a set computable in $\mathcal{A}$.

(2) Construct a nonstandard model $\mathcal{A}$ of PA such that there is no Solovay family $(t_n)_{n\geq 1}$ for Th($\mathcal{A}$) relative to $\mathcal{A}$ in which all the functions t_n change values at most once.

(3) Construct a nonstandard model $\mathcal{A}$ of PA such that there is no Solovay family $(t_n)_{n\geq 1}$ for Th($\mathcal{A}$) relative to $\mathcal{A}$ with a computable function f such that for each n, t_n changes values at most $f(n)$ times.

In the following sections, we answer each problem by constructing models with the required features. We therefore show that condition (b) of Solovay's Theorem may not be simplified in the three ways discussed above. Our solutions rely fundamentally on two new independence results that seem to be of a different sort from others currently in the literature. In the first independence result, Theorem 2.1, we show that for each n we can effectively find a Π_n sentence σ_n that is consistent over PA and any set of B_{n-1} sentences consistent with PA; we also have that $\neg\sigma_n$ is consistent with PA and any set of Σ_n sentences consistent with PA. We may thus find, for any completion T, completions T^* and T^{**}, so that $T \cap B_{n-1} \subseteq T^* \cap \Sigma_n \subseteq T^{**} \cap \Sigma_n$, where $\neg\sigma_n \in T^{**} - T^*$. The sentences σ_n are modifications of the Gödel-Rosser independent sentence [10]. We use this independence result to answer Problem 1. The second independence result, Theorem 2.3, iterates the first result; we use it to answer Problems 2 and 3. We also make the obvious extensions of the results to any consistent, computable extension of PA.

Here is the plan for the rest of the paper. We begin in Section 2 by stating and proving some independence results. In Section 3, we give our solution to Problem 1. The solution uses the first of the new independence results. In Section 4, we give our solution to Problem 2. Finally, in Section 5, we give our solution to Problem 3. The solutions to Problems 2 and 3 use the iterated independence result.

2. Independence results

In this section we state and prove the independence results that we will use in the following sections. Here is the first of these:

Theorem 2.1. *For every n, we can effectively find a sentence σ_n such that σ_n is Π_n, and the following hold:*

(a) *For any set Γ of B_{n-1} sentences such that* $\mathrm{PA} + \Gamma$ *is consistent,* $\mathrm{PA} + \Gamma + \sigma_n$ *is also consistent; and*

(b) *for any set Λ of Σ_n sentences such that* $\mathrm{PA} + \Lambda$ *is consistent,* $\mathrm{PA} + \Lambda + \neg\sigma_n$ *is also consistent.*

Proof. We begin by stating the Π_n sentence σ_n:

$\sigma_n := \forall p \forall \overline{u}[(p$ is a proof of me from PA + true Σ_{n-1} and Π_{n-1} sentences + one Σ_n sentence $\exists \overline{x} \chi(\overline{x})$ such that $\mathrm{Sat}_{\Pi_{n-1}}(\chi, \overline{u})) \to \exists q < p \exists \overline{u}' \leq \overline{u}(q$ is a proof of my negation from PA + true Σ_{n-1} and Π_{n-1} sentences + one true Σ_n sentence $\exists \overline{x} \chi'(\overline{x})$ such that $\mathrm{Sat}_{\Pi_{n-1}}(\chi', \overline{u}'))]$

Note that because of the explicit use of self-reference in σ_n, Gödel's fixed point lemma is needed to locate these sentences. Note also that the sentences make use of the predicates $\mathrm{Sat}_{\Sigma_n}(x, y)$ and $\mathrm{Sat}_{\Pi_n}(x, y)$. These predicates define satisfaction in PA for Σ_n and Π_n formulas, respectively. Thus $\mathrm{Sat}_{\Sigma_n}(x, y)$ expresses that the Σ_n formula $\varphi(y)$ coded by x is satisfied by y. Importantly, $\mathrm{Sat}_{\Sigma_n}(x, y)$ and $\mathrm{Sat}_{\Pi_n}(x, y)$ are Σ_n and Π_n formulas, respectively.

To see that σ_n is Π_n, note that it has the following logical form (ignoring bounded quantifiers):

$$\forall[(\Sigma_{n-1} \wedge \Pi_{n-1} \wedge \Pi_{n-1}) \to (\Sigma_{n-1} \wedge \Pi_{n-1} \wedge \Pi_{n-1})]$$

which is equivalent to

$$\forall[\neg(\Sigma_{n-1} \wedge \Pi_{n-1}) \vee (\Sigma_{n-1} \wedge \Pi_{n-1} \wedge \Pi_{n-1})]$$

which is equivalent to

$$\forall[\Pi_n \vee \Pi_n]$$

which is clearly Π_n.

Fix $n \geq 1$. First, we show that (a) holds. Let Γ be a set of B_{n-1} sentences such that $\mathrm{PA} + \Gamma$ is consistent. We want to show that $\mathrm{PA} + \Gamma + \sigma_n$ is also consistent. Suppose not; then $\mathrm{PA} + \Gamma \vdash \neg\sigma_n$. Consider a proof of $\neg\sigma_n$; it is a finite sequence of axioms of PA, B_{n-1} axioms (which we may replace by a set of Σ_{n-1} and Π_{n-1} axioms), and deductive consequences of these axioms. Let p^* be the code of this proof of $\neg\sigma_n$; it follows that p^* is standard. Let $\mathcal{A} \models \mathrm{PA} + \Gamma$; then $\mathcal{A}$ recognizes p^* as a proof of $\neg\sigma_n$. But $\mathcal{A} \models \neg\sigma_n$, so $\mathcal{A}$ witnesses that there is a tuple $\overline{u}$ and proof p of σ_n from PA + true Σ_{n-1} and Π_{n-1} sentences + one true Σ_n sentence $\exists \overline{x} \chi(\overline{x})$ such that $\mathrm{Sat}_{\Pi_{n-1}}(\chi, \overline{u})$, such that there do not exist $\overline{u}' \leq \overline{u}$ and $q < p$ where q is a proof of $\neg\sigma_n$ from the appropriate axioms. Since p^* is a proof of $\neg\sigma_n$ from PA plus just B_{n-1} sentences from Γ, p^* uses no Σ_n sentences and so we may take the witness $\overline{u}'$ to be $\emptyset$. Thus if $\neg\sigma_n$ is to hold in $\mathcal{A}$, it must be the case that either $p^* \not< p$ or $\emptyset \not\leq \overline{u}$. Since it must be the case (in a reasonable Gödel numbering) that for all $\overline{u}$, $\emptyset \leq \overline{u}$, the second condition cannot hold. If the first condition holds, $p^* \geq p$. But p^* is a standard number in $\mathcal{A}$, so p must also be a standard number in $\mathcal{A}$. Thus we have $\mathcal{A}$ witnessing that there are standard proofs of σ_n and $\neg\sigma_n$, a contradiction. Therefore $\mathrm{PA} + \Gamma + \sigma_n$ is consistent.

Next, we show that (b) holds. Let Λ be a set of Σ_n sentences such that $\mathrm{PA} + \Lambda$ is consistent. We want to show that $\mathrm{PA} + \Lambda + \neg\sigma_n$ is also consistent. Suppose not; then $\mathrm{PA} + \Lambda \vdash \sigma_n$. Let p^* be the (code of the) proof of σ_n. Consider this proof of σ_n; it is a finite sequence of axioms of PA, Σ_n axioms, and deductive consequences of these axioms. We may replace these finitely many Σ_n axioms with their conjunction. Note that this proof p^* is standard. Let $\mathcal{A} \models \mathrm{PA} + \Lambda$; then $\mathcal{A}$ recognizes p^* as a proof of σ_n. But $\mathcal{A} \models \sigma_n$, so $\mathcal{A}$ witnesses that there is a proof $q < p^*$ such that q is a proof of $\neg\sigma_n$ from the appropriate axioms. Since $q < p^*$ and p^* is standard, q is standard as well. Thus we have standard proofs of σ_n and $\neg\sigma_n$ in $\mathcal{A}$, a contradiction. Therefore $\mathrm{PA} + \Lambda + \neg\sigma_n$ is consistent.

□

Taking the same argument and applying it to any consistent, computable extension of PA rather than just to PA, we get the following result:

Theorem 2.2. *Let P be a consistent, computable set of sentences including the axioms of PA. For every n, we can effectively find a sentence σ_n^P such that σ_n^P is Π_n, and the following hold:*

(a) *For any set Γ of B_{n-1} sentences such that $P + \Gamma$ is consistent, $P + \Gamma + \sigma_n^P$ is also consistent; and*

(b) *for any set Λ of Σ_n sentences such that $P + \Lambda$ is consistent, $P + \Lambda + \neg\sigma_n^P$ is also consistent.*

This result follows by modifying the sentences σ_n of Theorem 2.1 to refer to the sentences in P rather than to the axioms of PA.

Next, we extend these results to get k many Π_n sentences $\sigma_n^1, \ldots, \sigma_n^k$, for any k, with independence properties like those of Theorems 2.1 and 2.2. We begin with the following extension:

Theorem 2.3. *For every n and for every $k \geq 1$, we can effectively find a sequence of sentences σ_n^1, σ_n^2, $\ldots$, σ_n^k such that each σ_n^i is Π_n, and the following hold:*

(a) *For any set Γ of B_{n-1} sentences such that $\mathrm{PA} + \Gamma$ is consistent, the collection $\mathrm{PA} + \Gamma + \sigma_n^1 + \cdots + \sigma_n^k$ is also consistent.*

(b) *For any set Λ of Σ_n sentences and any $1 \leq i \leq k$ such that $\mathrm{PA} + \Lambda + \sigma_n^i + \cdots + \sigma_n^k$ is consistent, the set $\mathrm{PA} + \Lambda + \neg\sigma_n^i + \sigma_n^{i+1} + \cdots + \sigma_n^k$ is also consistent.*

Proof. We repeatedly iterate Theorem 2.2 to prove both (a) and (b). Let n and k be fixed. Let σ_n^k be the sentence σ_n^{PA} given by Theorem 2.2. For each $1 \leq i < k$, we define the set $P_i = \mathrm{PA} + \sigma_n^{i+1} + \cdots + \sigma_n^k$.

We prove parts (a) and (b) inductively. Suppose first that $\mathrm{PA} + \Gamma + \sigma_n^{i+1} + \cdots + \sigma_n^k$ is consistent. Apply Theorem 2.2(a) to the set P_i defined above to get a sentence $\sigma_n^{P_i}$ such that $P_i + \Gamma + \sigma_n^{P_i}$ is consistent. Let $\sigma_n^i = \sigma_n^{P_i}$; then we have shown that $\mathrm{PA} + \Gamma + \sigma_n^i + \sigma_n^{i+1} + \cdots + \sigma_n^k$ is consistent. Thus (a) holds. Suppose $\mathrm{PA} + \Lambda + \sigma_n^i + \cdots + \sigma_n^k$ is consistent. Apply Theorem 2.2(b) to P_i to get that $P_i + \Lambda + \neg\sigma_n^i$ is consistent; or, equivalently, that $\mathrm{PA} + \Lambda + \neg\sigma_n^i + \sigma_n^{i+1} + \cdots + \sigma_n^k$ is consistent. Thus (b) holds. □

We also have the following, slightly more general result:

Theorem 2.4. *Let P be a consistent, computable set of sentences including the axioms of PA. For every n and every $k \geq 1$, we can effectively find a sequence of sentences σ_n^1, σ_n^2, $\ldots$, σ_n^k such that each σ_n^i is Π_n, and the following hold:*

(a) *For any set Γ of B_{n-1} sentences such that $P + \Gamma$ is consistent, the collection $P + \Gamma + \sigma_n^1 + \cdots + \sigma_n^k$ is also consistent.*

(b) *For any set Λ of Σ_n sentences and any $1 \leq i \leq k$ such that $P + \Lambda + \sigma_n^i + \cdots + \sigma_n^k$ is consistent, the set $P + \Lambda + \neg\sigma_n^i + \sigma_n^{i+1} + \cdots + \sigma_n^k$ is also consistent.*

3. Solution to Problem 1

In this section, we present our solution to Problem 1. In Theorem 3.2, we construct a nonstandard model $\mathcal{A}$ of PA such that there is no Solovay family $(t_n)_{n\geq 1}$ for $\mathrm{Th}(\mathcal{A})$ relative to $\mathcal{A}$ in which all the functions t_n are constant.

We make use of the following notation here and in upcoming sections. First, for any completion T of PA and any n, let $\widetilde{T_n} = T_n + \{\neg\varphi : \varphi \text{ is } \Sigma_n, \varphi \notin T_n\}$. Second, for a formula φ that is Π_n in prenex normal form, let $\mathrm{neg}(\varphi)$ be the natural Σ_n formula that is logically equivalent to $\neg\varphi$. Third, let $\varphi_e^{\Delta^0_m(X)}(x)$ denote the (code for the) e^{th} computable function using oracle $\Delta^0_m(X)$ for some set X.

Fix a Scott set $\mathcal{S}$ and an enumeration R of $\mathcal{S}$. By Marker's Theorem 1.7, we may take R to be an effective enumeration. Our plan to solve Problem 1 is as follows. First, we determine a completion T of PA and a Solovay family $(t_n)_{n\geq 1}$ for T relative to R with some special features. We then apply Solovay's Theorem 1.10 to get a model $\mathcal{A} \models T$ answering Problem 1 affirmatively. Here is the result giving the completion and Solovay family:

Lemma 3.1. *There exists a completion T of* PA *with a Solovay family $(t_n)_{n\geq 1}$ for T relative to R such that for all n, $\varphi_n^{\Delta^0_n(R)}(n)$ is not an index for $T \cap \Sigma_n$ as a set computable in R.*

Proof. Start with $T_0 = \mathrm{PA} \cap \Sigma_0$. We define T_n and the Solovay family $(t_n(s))_{n\geq 1}$ such that t_n is $\Delta^0_n(R)$ uniformly in n and the following requirements are met:

Limit$_n$: $\lim_{s\to\infty} t_n(s) = t_n^*$, where $R_{t_n^*} = T_n$

Subset$_n$: For all $s, t_n(s)$ is an R-index for a subset of T_n.

Diagonal$_n$: If $\varphi_n^{\Delta^0_n(R)}(n) \downarrow = e$, then $\varphi_e^R \neq \chi_{T_n}$.

We choose T_n and t_n inductively. Recall that $t_n(s)$ approximates indices for true fragments of T_n for varying s and yields the index for T_n in its entirety in the limit. Suppose we have already determined T_{n-1} and t_{n-1}. Here is how we determine T_n and t_n. Using $\Delta^0_n(R)$, compute $a = \lim_{s\to\infty} t_{n-1}(s)$, so that $R_a = T_{n-1}$. Using the index a, we may effectively find an R-index for a completion T^* of $\mathrm{PA} \cup \widetilde{T_{n-1}} \cup \{\sigma_n\}$, where σ_n comes from Theorem 2.1(a). We then compute an index i^* for $T_n^* = T^* \cap \Sigma_n$. For the time being, define T_n as T_n^*. We wait for a stage s where $\varphi_{n,s}^{\Delta_n^0(R)}(n) \downarrow = e$ and $\varphi_e^R(\text{neg}(\sigma_n)) = 0$.

Case 1: We reach such a stage s.

To meet requirement Diagonal_n, we switch our choice of T_n as follows. Using Theorem 2.1(b) and letting $\Lambda = T_n^*$, we effectively find an index for a completion T^{**} of the set $\text{PA} \cup T_n^* \cup \{\neg\sigma_n\}$. We then compute an index i^{**} for $T^{**} \cap \Sigma_n$. Redefine T_n as T_n^{**}. For stages $r < s$, define $t_n(r) = i^*$. For $r \geq s$, define $t_n(r) = i^{**}$. Note that $T_n^* \subseteq T_n^{**}$. Thus we have met requirement Subset_n. We have also insured that t_n is defined for all s and is eventually constant, demonstrating that we meet requirement Limit_n.

Case 2: We have not yet reached such a stage s.

In this case, requirement Diagonal_n is not yet threatened. If we find such a stage s, we enter Case 1. As long as we do not, we let $T_n = T_n^*$ and let $t_n(r) = i^*$ for all r. As long as we stay in this case, requirements Limit_n and Subset_n are clearly met. □

The following result answers Problem 1 affirmatively:

Theorem 3.2. *There exists a nonstandard model $\mathcal{A}$ of* PA *such that if $T = Th(\mathcal{A})$, there is no e such that for all n, $\varphi_e^{\Delta_n^0(\mathcal{A})}(n)$ is an index for $T \cap \Sigma_n$ as a set computable in $\mathcal{A}$.*

Proof. Let T be the completion of PA with Solovay family $(t_n)_{n\geq 1}$ that is given by Lemma 3.1. Use Solovay's Theorem 1.10 to get a model $\mathcal{A} \models T$ with $\mathcal{A} \leq_T R$. Suppose there is some e such that for all n, $\varphi_e^{\Delta_n^0(\mathcal{A})}(n)$ is an index for $T \cap \Sigma_n$ as a set computable in $\mathcal{A}$. Then there is some e^* such that for all n, $\varphi_{e^*}^{\Delta_n^0(R)}(n) = \varphi_e^{\Delta_n^0(\mathcal{A})}(n)$. This contradicts the choice of T. □

4. Solution to Problem 2

In this section, we present our solution to Problem 2. In Theorem 4.2, we construct a nonstandard model $\mathcal{A}$ of PA such that there is no Solovay family $(t_n)_{n\geq 1}$ for $\text{Th}(\mathcal{A})$ relative to $\mathcal{A}$ in which all the functions t_n change values at most once. In fact, for each k, we construct a model $\mathcal{A}$ with no Solovay family for $\mathrm{Th}(\mathcal{A})$ relative to $\mathcal{A}$ in which all the functions t_n change values at most k many times.

As before, fix a Scott set $\mathcal{S}$ and an effective enumeration R of $\mathcal{S}$. Our plan to solve Problem 2 is as follows. We specify T and $(t_n)_{n\geq 1}$, then apply Solovay's Theorem 1.10 to get a model $\mathcal{A} \models T$ answering Problem 2 affirmatively. Here is the result giving the completion and Solovay family:

Lemma 4.1. *For every k, there exists a completion T of* PA *with a Solovay family $(t_n)_{n\geq 1}$ for T relative to R such that for all $n > 1$, one of the following holds:*

1. *It is not the case that $\lim_{s\to\infty} \varphi_n^{\Delta_n^0(R)}(n,s) \downarrow= i$, where $R_i = T_n$.*
2. *There exists s such that $\varphi_n^{\Delta_n^0(R)}(n,s) \downarrow= i$, where $R_i \not\subseteq T_n$.*
3. *There exist $j_1 \neq j_2 \neq \cdots \neq j_k$ and $s_1, \ldots, s_k$ such that for $\ell = 1, \ldots, k$, $\varphi_n^{\Delta_n^0(R)}(n, s_\ell) = j_\ell$ and $T_n \notin \{R_{j_1}, \ldots, R_{j_k}\}$.*

Proof. Fix k. Start with $T_0 = \mathrm{PA} \cap \Sigma_0$. We begin with T_1 given as the Σ_1 part of an arbitrary completion of T_0 computable in R. We also begin with t_1 given, as a function computable in $\Delta_1^0(R)$. We will define $T_n = T \cap \Sigma_n$ and the Solovay family $(t_n(s))$ for $n \geq 2$ such that t_n is $\Delta_n^0(R)$ uniformly in n, and the following requirements are met:

Limit$_n$: $\lim_{s\to\infty} t_n(s) = t_n^*$, where $R_{t_n^*} = T_n$

Subset$_n$: For all $s, t_n(s)$ is an R-index for a subset of T_n.

Changes$_n$: If $\lim_{s\to\infty} \varphi_n^{\Delta_n^0(R)}(n,s)$ is an R-index for T_n, then either there exists s such that $\varphi_n^{\Delta_n^0(R)}(n,s)$ is not an R-index for a subset of T_n, or else there exist $j_1 \neq j_2 \neq \cdots \neq j_k$ and $s_1, \ldots, s_k$ such that for $\ell = 1, \ldots, k$, $\varphi_n^{\Delta_n^0(R)}(n, s_\ell) = j_\ell$ and $T_n \notin \{R_{j_1}, \ldots, R_{j_k}\}$.

We choose T_n and t_n inductively. Suppose we have already determined T_{n-1} and t_{n-1}, for some $n \geq 2$. Here is how we determine T_n and t_n. Using $\Delta_n^0(R)$, compute $a = \lim_{s\to\infty} t_{n-1}(s)$, so that $R_a = T_{n-1}$. We may then find a completion T^1 of $\mathrm{PA} \cup \widetilde{T_{n-1}} \cup \{\sigma_n^1, \sigma_n^2, \ldots, \sigma_n^k\}$, using Theorem 2.3(a). We may also effectively find an R-index for T^1, using our effective enumeration R. We then compute an R-index i_1 for the fragment $T_n^1 = T^1 \cap \Sigma_n$. For the time being, define T_n as T_n^1. Define $t_n(s) = i_1$

for all stages s until we reach the first stage r_1 such that for some $s \leq r_1$ and some j_1, $\varphi_{n,r_1}^{\Delta_n^0(R)}(n,s) \downarrow= j_1$ and $\text{neg}(\sigma_n^1) \notin R_{j_1}$.

If this never happens, it never happens for one of two reasons. The first is that there is no stage r_1 so that for some $s \leq r_1$, $\varphi_{n,r_1}^{\Delta_n^0(R)}(n,s) \downarrow$. The second is that $\varphi_{n,r_1}^{\Delta_n^0(R)}(n,s) \downarrow= j_1$ but $\text{neg}(\sigma_n^1) \in R_{j_1}$. If the former is the case, then we will have met condition (1) of the lemma. If the latter is the case, then $R_{j_1} \not\subseteq T_n^1 = T_n$. Thus we will have met condition (2) of the lemma. In both cases, we have met requirements Limit$_n$ and Subset$_n$ by our definitions of T_n and t_n. We have also trivially met requirement Changes$_n$.

If, on the other hand, we reach the first stage r_1 such that for some $s \leq r_1$ and some j_1, $\varphi_{n,r_1}^{\Delta_n^0(R)}(n,s) \downarrow= j_1$ and $\text{neg}(\sigma_n^1) \notin R_{j_1}$, then $\varphi_n^{\Delta_n^0(R)}(n,s)$ threatens requirement Changes$_n$ by appearing to converge to an R-index for T_n. To avert this threat, we switch our choice of T_n as follows. Using Theorem 2.3(b), we effectively find a completion T^2 of $\text{PA} \cup \widetilde{T_n^1} \cup \{\neg\sigma_n^1, \sigma_n^2, \ldots, \sigma_n^k\}$. We then compute an index i_2 for $T^2 \cap \Sigma_n$. Redefine T_n as T_n^2. For stages $s \geq r_1$, define $t_n(s) = i_2$, until we reach the first stage r_2 such that for some $s \leq r_2$ and some j_2, $\varphi_{n,r_2}^{\Delta_n^0(R)}(n,s) \downarrow= j_2$, $\text{neg}(\sigma_n^1) \in R_{j_2}$, and $\text{neg}(\sigma_n^2) \notin R_{j_2}$.

As before, if this never happens then we have met all our requirements. If it does happen, then to meet requirement Changes$_n$ we again switch our choice of T_n as follows. Using Theorem 2.3(b), we find a completion T^3 of $\text{PA} \cup \widetilde{T_n^2} \cup \{\neg\sigma_n^2, \sigma_n^3, \ldots, \sigma_n^k\}$. We then compute an index i_3 for $T^3 \cap \Sigma_n$. Redefine T_n as T_n^3. For stages $s \geq r_2$, define $t_n(s) = i_3$, until we reach the first stage r_3 such that for some $s \leq r_3$ and some j_3, $\varphi_{n,r_3}^{\Delta_n^0(R)}(n,s) \downarrow= j_3$, $\text{neg}(\sigma_n^1) \in R_{j_3}$, $\text{neg}(\sigma_n^2) \in R_{j_3}$, and $\text{neg}(\sigma_n^3) \notin R_{j_3}$. We continue challenging $\varphi_n^{\Delta_n^0(R)}(n,s)$ to change values like this for up to k many times. If $\varphi_n^{\Delta_n^0(R)}(n,s)$ fails to meet one of our challenges, then as before we will have met all of our requirements and may stop our challenges.

For our $m < k^{th}$ challenge, we ask $\varphi_n^{\Delta_n^0(R)}(n,s)$ to match T_n^m. The fragment T_n^m has the feature that for all $1 \leq q < m$, $\text{neg}(\sigma_n^q) \in T_n^m$, but $\text{neg}(\sigma_n^m) \notin T_n^m$. We keep $t_n(s) = i_m$ for all $s \geq r_{m-1}$ until we reach the first stage r_m such that for some $s \leq r_m$ and some j_m, the following hold:

(i) $\varphi_{n,r_m}^{\Delta_n^0(R)}(n,s) \downarrow= j_m$

(ii) For each $1 \leq q < m$, $\text{neg}(\sigma_n^q) \in R_{j_m}$

(iii) $\text{neg}(\sigma_n^m) \notin R_{j_m}$

Again, if we find no stage r_m meeting these conditions, then we have met all our requirements and can stop challenging $\varphi_n^{\Delta_n^0(R)}(n, s)$ to change values. If we do find such a stage, then to meet requirement Changes_n we switch our choice of T_n as follows. We find a completion T^{m+1} of $\text{PA} \cup \widetilde{T_n^m} \cup \{\neg\sigma_n^m, \sigma_n^{m+1}, \ldots, \sigma_n^k\}$. We then compute an index i_{m+1} for $T^{m+1} \cap \Sigma_n$. Redefine T_n as T_n^{m+1}. Define $t_n(s) = i_{m+1}$ for stages $s \geq r_m$, until we reach a stage r_{m+1} where $\varphi_n^{\Delta_n^0(R)}(n, s)$ appears to return an index for T^{m+1} as in (i),(ii), and (iii) above. If we never reach such a stage, then as above we will have met each of our requirements. We continue challenging $\varphi_n^{\Delta_n^0(R)}(n, s)$ this way until we reach $m = k$. We then issue $\varphi_n^{\Delta_n^0(R)}(n, s)$ one last challenge. We wait for the first stage r_k such that for some $s \leq r_k$ and some j_k, the following hold:

(i) $\varphi_{n,r_k}^{\Delta_n^0(R)}(n, s) \downarrow = j_k$

(ii) For each $1 \leq q < k$, $\text{neg}(\sigma_n^q) \in R_{j_k}$

(iii) $\text{neg}(\sigma_n^k) \notin R_{j_k}$

If we find no stage r_k meeting these conditions, then we have met all our requirements. If we do find such a stage, then to meet requirement Changes_n we switch our choice of T_n one last time as follows. We find a completion T^{k+1} of $\text{PA} \cup \widetilde{T_n^k} \cup \{\neg\sigma_n^k\}$. We then compute an index i_{k+1} for $T^{k+1} \cap \Sigma_n$. Redefine T_n as T_n^{k+1}. Define $t_n(s) = i_{k+1}$ for all stages $s \geq r_k$.

If all k many challenges are met, then we may stop changing T_n. We will have that $T_n^m \subseteq T_n^{k+1}$ for $1 \leq m \leq k$. Thus we have met requirement Subset_n. We have defined t_n for all s such that $t_n(s)$ is eventually constant. Thus we have met requirement Limit_n. Finally, we have $j_1 \neq j_2 \neq \cdots \neq j_k$ and $r_1, \ldots, r_k$ such that for $\ell = 1, \ldots, k$, $\varphi_n^{\Delta_n^0(R)}(n, r_\ell) = j_\ell$ and $T_n \notin \{R_{j_1}, \ldots, R_{j_k}\}$. We have thus also met requirement Changes_n. □

We can now answer Problem 2 with the following result:

Theorem 4.2. *For every k, there exists a nonstandard model $\mathcal{A}$ of* PA *such that if $T = Th(\mathcal{A})$, there is no e with the following features for all n:*

(a) $\lim_{s\to\infty} \varphi_e^{\Delta_n^0(\mathcal{A})}(n,s)$ *is an index for T_n, as a set computable in $\mathcal{A}$;*
(b) *For all s, $\varphi_e^{\Delta_n^0(\mathcal{A})}(n,s)$ is an index for a subset of T_n, as a set computable in $\mathcal{A}$;*
(c) *The set $\{\varphi_e^{\Delta_n^0(\mathcal{A})}(n,s) : s \in \omega\}$ has cardinality less than or equal to k.*

Proof. Fix k (for the exact statement of Problem 2, use $k = 1$). Let T be the completion of PA with Solovay family $(t_n)_{n\geq 1}$ given by Lemma 4.1. Use Solovay's Theorem 1.10 to get a model $\mathcal{A} \models T$ with $\mathcal{A} \leq_T R$. Suppose there is some e for which (a), (b), and (c) of the theorem hold for all n. Then there is some e^* such that for all n and s, $\varphi_{e^*}^{\Delta_n^0(R)}(n,s) = \varphi_e^{\Delta_n^0(\mathcal{A})}(n,s)$. By padding, we may take $e^* > 1$. Note that since (a) and (b) hold, the completion T given by Lemma 4.1 satisfies condition (3). By (c), it follows that for all n, $|\{\varphi_n^{\Delta_n^0(R)}(n,s) : s \in \omega\}| \leq k$. Let $n = e^*$. This contradicts condition (3) of the lemma. □

5. Solution to Problem 3

In this section, we present our solution to Problem 3. In Theorem 5.2, we construct a nonstandard model $\mathcal{A}$ of PA such that there is no Solovay family $(t_n)_{n\geq 1}$ for Th($\mathcal{A}$) relative to $\mathcal{A}$ with a computable function f such that $t_n(s)$ changes values at most $f(n)$ times.

As before, fix a Scott set $\mathcal{S}$ and an effective enumeration R of $\mathcal{S}$. We proceed as we did with Problem 2. We specify T and $(t_n)_{n\geq 1}$, then apply Solovay's Theorem 1.10 to get a model $\mathcal{A} \models T$ answering Problem 3 affirmatively. Here is the result giving the completion and Solovay family:

Lemma 5.1. *There exists a completion T of* PA *with a Solovay family $(t_n)_{n\geq 1}$ for T relative to R such that for all $n = \langle e_1, e_2\rangle$ such that $n > 2$, if $\varphi_{e_2}(n) \downarrow= k$, one of the following holds:*

(1) *It is not the case that $\lim_{s\to\infty} \varphi_{e_1}^{\Delta_n^0(R)}(n,s) \downarrow= i$, where $R_i = T_n$.*
(2) *There exists s such that $\varphi_{e_1}^{\Delta_n^0(R)}(n,s) \downarrow= i$ and $R_i \not\subseteq T_n$.*

(3) *There exist* $j_1 \neq j_2 \neq \cdots \neq j_k$ *and* $s_1, \ldots, s_k$ *such that for* $\ell = 1, \ldots, k$, $\varphi_{e_1}^{\Delta_n^0(R)}(n, s_\ell) = j_\ell$ *and* $T_n \notin \{R_{j_1}, \ldots, R_{j_k}\}$.

Proof. We describe the idea of the proof, which is essentially a modified version of the proof of Lemma 4.1. We begin with the fragments T_1 and T_2 already determined, as well as the corresponding functions t_1 and t_2. We decode each n as a pair $\langle e_1, e_2 \rangle$. The second element of the pair, e_2, is an index for a computable function φ_{e_2}. The first element, e_1, is an index for a binary computable function relative to $\Delta_n^0(R)$, namely $\varphi_{e_1}^{\Delta_n^0(R)}(n, s)$. By starting with the $\Delta_2^0(R)$ function t_2 given, we may consult $\Delta_2^0(R)$ to check if $\varphi_{e_2}(m) \downarrow$.

As in the proof of Lemma 4.1, we want to challenge $\varphi_{e_1}^{\Delta_n^0(R)}(n, s)$ to change values a number of times. For each n, however, the number of changes needed depends upon the value of $\varphi_{e_2}(n)$. Using $\Delta_2^0(R)$, we may decide if $\varphi_{e_2}(n) \downarrow$. If it halts, we wait until we reach the first stage where it halts and then compute its value $k = \varphi_{e_2}(n)$. We then proceed as in the proof of Lemma 4.1, challenging $\varphi_{e_1}^{\Delta_n^0(R)}(n, s)$ to change values k many times.

We now proceed more formally. Start with $T_0 = \mathrm{PA} \cap \Sigma_0$. As we said above, we begin with T_1 and T_2, as well as t_1 and t_2, given. We will define T_n and the Solovay family $(t_n(s))$ such that each t_i is $\Delta_i^0(R)$ uniformly in i, and the following requirements are met for each $n = \langle e_1, e_2 \rangle$ such that $\varphi_{e_2}(n) \downarrow$ with output k:

Limit_n : $\lim_{s \to \infty} t_n(s) = t_n^*$, where $R_{t_n^*} = T_n$

Subset_n : For all $s, t_n(s)$ is an R-index for a subset of T_n.

$\mathrm{Changes}_n$: If $\lim_{s \to \infty} \varphi_{e_1}^{\Delta_n^0(R)}(n, s)$ is an R-index for T_n, then either there exists s such that $\varphi_{e_1}^{\Delta_n^0(R)}(n, s)$ is not an R-index for a subset of T_n, or else there exist $j_1 \neq j_2 \neq \cdots \neq j_k$ and $s_1, \ldots, s_k$ such that for $\ell = 1, \ldots, k$, $\varphi_{e_1}^{\Delta_n^0(R)}(n, s_\ell) = j_\ell$ and $T_n \notin \{R_{j_1}, \ldots, R_{j_k}\}$.

Suppose we have already determined T_{n-1} and t_{n-1}. Here is how we determine T_n and t_n, in an attempt to meet requirements Limit_n, Subset_n, and $\mathrm{Changes}_n$. We first decode $n = \langle e_1, e_2 \rangle$. Next, we use $\Delta_n^0(R)$ to decide if $\varphi_{e_2}(n) \downarrow$. If it halts, we wait until we reach the first stage where it halts and compute its value $k = \varphi_{e_2}(n)$. If it does not halt, then the lemma trivially holds for n. Supposing it does halt, we then proceed as we did in the proof of Lemma 4.1, issuing $\varphi_{e_1}^{\Delta_n^0(R)}(n,s)$ k-many challenges to match potential candidates $T_n^1, T_n^2, \ldots, T_n^{k+1}$ for the fragment T_n. □

We can now answer Problem 3 with the following result:

Theorem 5.2. *There exists a nonstandard model $\mathcal{A}$ of* PA *such that if $T = Th(\mathcal{A})$, there do not exist e_1 and e_2 with the following features for all n:*

(a) $\lim_{s\to\infty} \varphi_{e_1}^{\Delta_n^0(\mathcal{A})}(n,s)$ *is an index for T_n, as a set computable in $\mathcal{A}$;*
(b) *For all s, $\varphi_{e_1}^{\Delta_n^0(\mathcal{A})}(n,s)$ is an index for a subset of T_n, as a set computable in $\mathcal{A}$;*
(c) *The set $\{\varphi_{e_1}^{\Delta_n^0(\mathcal{A})}(n,s) : s \in \omega\}$ has cardinality less than or equal to $\varphi_{e_2}(n)$.*

Proof. Let T be the completion of PA with Solovay family $(t_n)_{n\geq 1}$ given by Lemma 5.1. Use Solovay's Theorem 1.10 to get a model $\mathcal{A} \models T$ with $\mathcal{A} \leq_T R$. Suppose there exist e_1, e_2 for which (a), (b), and (c) of the theorem hold for all n. It follows that there exists e_1^* such that for all n and s, $\varphi_{e_1^*}^{\Delta_n^0(R)}(n,s) = \varphi_{e_1}^{\Delta_n^0(\mathcal{A})}(n,s)$. By padding, we may take $e_1^* > 2$. Note that since (a) and (b) hold, the completion T given by Lemma 5.1 satisfies condition (3). By (c), it follows that for all n, $|\{\varphi_{e_1}^{\Delta_n^0(R)}(n,s) : s \in \omega\}| \leq \varphi_{e_2}(n)$. Let $n = e_1^*$. This contradicts condition (3) of the lemma. □

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